Criticality in a dynamic mixed system

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(Received 6 July 2000; revised manuscript received 6 June 2001; published 25 October 2001)

We suggest a dynamic generalization of the simplest static hierarchical mixed model introduced by Shnirman and Blanter [Phys. Rev. Lett. **81**, 5445 (1998); Phys. Rev. E. **60**, 5111 (1998). We show that the stationary solution of the dynamic mixed model (DMM) demonstrates, in general, a linear form of the magnitude-frequency relation and may be considered a self-organized critical system. The dynamic mixed model demonstrates three principal kinds of system behavior: stability, catastrophe, and scale invariance. We show that the catastrophic area exists for all parameters of the mixture, and obtain three analytical expressions for boundary conditions of the stability and the scale invariance domains. As in the static model scale invariance appears as a result of a strong heterogeneity of the mixture. We describe how the magnitude-frequency relation reflects parameters of the heterogeneity and healing conditions for different domains of system behavior. Deviation of the DMM from the static mixed model and possible applications to earthquake prediction are discussed.

DOI: 10.1103/PhysRevE.64.056123 PACS number(s): 05.40.−a, 91.30.Bi

I. INTRODUCTION

Considering self-similar hierarchical models we are interested in two problems: the origin of self-organized criticality (SOC) and its possible relation with prognostic features of the system. Besides the theoretical interest these problems have wide applications in geophysics, in particular in earthquake prediction. The well-known Gutenberg-Richter law establishes the linear form of the magnitude-frequency relation for earthquakes [3]. It is a basic statistical relation of seismology usually associated with the self-organized criticality phenomenon [4,5], however, its origin in seismicity is not completely clear. Statistical observations show that the slope of Gutenberg-Richter law is different for different seismoactive regions and period of time [6]. The slope for main shocks differs from the slope when aftershocks are included. The slope for aftershocks series differs from the slope of foreshock series. The unity slope was obtained only for the average world seismicity [7]. Thus, the criticality of seismicity is more complicated than the self-organized criticality realized, for example, in the sand-pile model [4]. It was expected that the slope of Gutenberg-Richter law may be related with fractal scaling properties of the system of faults, however, it is not proved [8].

Scaling properties of seismicity are related with the earthquake prediction: variations of the slope and the upward-downward bend of the magnitude-frequency relation can be successfully used for the earthquake prediction [9,10]; several algorithms of earthquake prediction use the variation of seismic activity as a prognostic functional [11,12] that may be also reformulated in terms of variation of the magnitude-frequency relation. The exact knowledge of the nature of scaling laws in seismicity is not necessary for the application to the earthquake prediction but may be useful for the improving of existing algorithm as well as for the understanding of the origin of successes and fails. In the present paper we investigate scaling properties of the dynamic mixed model (DMM) and try to relate them with predictability of strong events extending results obtained for the static one [13].

Stable critical behavior is obtained for different models referred to as self-organized critical systems. For some of them the origin of SOC is understood [5,14]. A lot of work are recently devoted to investigations of the origin of SOC in sand-pile models [4] and describe possible ways to obtain deviations from criticality for such systems [15]. The self-organized criticality may be also obtained as a result of a feedback relation that attracts the system to the critical point and governs a linear form of the magnitude-frequency relation in average [9,16]. In [1] we have suggested a static hierarchical mixed model and demonstrated that the self-organized criticality appears as a result of strong heterogeneity of destruction conditions. However, the static model is slightly related with seismic process. Now we suggest the dynamic model and show that the heterogeneity determines criticality as a general form of system behavior. In particular, scale invariance appears in the DMM as a result of strong heterogeneity of the resistance to the stress.

Critical behavior of the system is indicated by a linear magnitude-frequency relation. The slope of the magnitude-frequency relation may be constant in time, like it is for the sand-pile model [4] or not, like in the case of the attractive point [9]. The origin of different slopes of the magnitude-frequency relation in seismicity is not more clear than the origin of criticality itself. Assuming that the heterogeneity is the origin of SOC, it is easy to obtain different slopes depending on parameters of heterogeneity [2]. Below we show that for the stationary solution of the DMM the slope of magnitude-frequency relation reflects parameters of heterogeneity, like it was for the static model, but it also depends on relaxing features of the system.

Predictability of strong events in SOC models was considered by Pepke and Carlson [17]. They have shown that different predictability of strong events may be obtained applying the same algorithm of prediction to different SOC systems. Self-organized criticality of the sand-pile model [4]

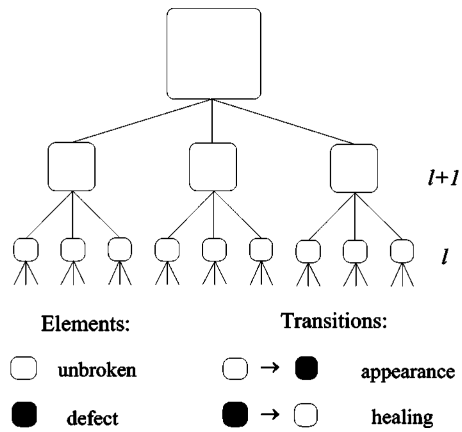

FIG. 1. Hierarchical system with branching number $n=3$.

is characterized by the bad predictability of events of large size [17]. When the linear form of the magnitude-frequency relation is a result of attractive critical point, then the magnitude-frequency relation depends on time and its temporal variation allows to predict strong events in the system [9]. In the static hierarchical mixed model [2] the change of system parameters is reflected in the magnitude-frequency relation that allows to predict the increasing probability of strong events [13]. For the present dynamic mixed model we show how relaxing features of the system affect prognostic properties of large-size events.

We describe the model in Sec. II and its behavior in general case in Sec. III. The slope of magnitude-frequency relation is derived from system equations for different domains of model behavior in Sec. IV. Boundary conditions of the scale invariance domain are obtained in Sec. V. Section VI contains conclusions and discussion of obtained results in respect to the earthquake prediction.

II. MODEL

We suggest a hierarchical model — DMM — that may be considered as a generalization both of static mixed model [1,2] and of the homogeneous hierarchical model of defect development [18].

We consider a hierarchical model with branching number $n=3$ and two possible state of elements: broken (defects) and unbroken one (Fig. 1). Evolution of the system is governed by two opposite process: appearance of new defects and healing of old defects. An unbroken element transit to the defect state, when a critical configuration of defects appears in the relevant group of three elements of the previous level. This rule determines the inverse cascade of destruction in the model.

A. Heterogeneity

In the homogeneous model [18] there was the same critical configuration for all elements of the system: two or more defects in a group of three elements determine the defect state of the relevant element of the superior level. Following [1,2] we assume that there are three kinds of elements in the system: the critical configuration for kth kind of element contains k or more defects in the relevant group of the inferior level. Different kinds of elements are mixed with concentrations a_k, where $a_1+a_2+a_3=1$. We assume self-similarity of the mixture, therefore, concentrations a_k are the same for all levels of the system. A density of defects of kth kind at level l in time t is denoted as $p_k(l,t)$, then the total density of defects is expressed as a weighted sum of p_k

$$p(l,t)=a_1p_1(l,t)+a_2p_2(l,t)+a_3p_3(l,t). \tag{1}$$

B. Relaxation

The healing process determines the transition of broken elements (defects) back to the unbroken state. The healing is similar for all kinds of elements and depends only on the scale level. The healing intensity $\beta(l)$ at level l is defined as follows:

$$\beta(l)=\beta_0 c^l. \tag{2}$$

Parameters of healing govern relaxing properties of the system.

C. Kinetic equations

Unlike homogeneous system, the present model has three kinds of appearance intensity $\alpha_k(l,t)$ relevant to each kind k of element, kinetic equations may be written for each kind of element

$$p_k(l,t+1)=p_k(l,t)[1-\beta(l)]+[1-p_k(l,t)]\alpha_k(l,t). \tag{3}$$

The second term in right side of Eq. (3) denotes the density of new defects, referred to as events, so the density of events of kth kind $q_k(l,t)$ is expressed as follows:

$$q_k(l,t+1)=[1-p_k(l,t)]\alpha_k(l,t). \tag{4}$$

The density of all events at level l in time t is a weighted sum of $q_k(l,t)$

$$q(l,t)=a_1q_1(l,t)+a_2q_2(l,t)+a_3q_3(l,t). \tag{5}$$

A kinetic equation for the density of all kinds of defects is

$$p(l,t+1)=p(l,t)[1-\beta(l)]+[1-p(l,t)]\alpha(l,t), \tag{6}$$

where $\alpha(l,t)$ is by definition the appearance intensity of new defects.

D. Appearance of new defects

The value of the appearance intensity $\alpha(l,t)$ may be obtained by substitution of Eqs. (1) and (3) into Eq. (6), and it satisfies to the following equation:

$$\alpha(l,t)[1-p(l,t)]=\sum_{k=1}^{3} a_k\alpha_k(l,t)[1-p_k(l,t)]. \tag{7}$$

The bottom level of the system differs from others, because defects randomly appears at this level and all elements are similar. Appearance intensity of new defects at the bottom level is assumed to be constant $\alpha(1,t)=\alpha_0$.

It follows from Eqs. (4), (5), and (7) that the density of events is

$$q(l,t)=[1-p(l,t)]\alpha(l,t). \tag{8}$$

Appearance intensities of new defects $\alpha_k(l+1,t)$ at level $l+1$ are determined by the conditional probability of critical configurations of the previous level l provided that a new defect can appear in a group of three elements of level l. The latter condition establishes a relation between defects of two consequent levels: the probability of an unbroken element relevant to a group of three defects is assumed negligible.

We assume that different kinds of elements of two consequent levels are distributed independently, therefore, appearance intensities of the superior level $\alpha_k(l+1,t)$ are calculated using the appearance intensity $\alpha(l,t)$ and the density of defects $p(l,t)$ of the previous level

$$\alpha_1(l+1,t)=\frac{[1-p(l,t)]^3[\alpha^3(l,t)+3\alpha^2(l,t)\{1-\alpha(l,t)\}+3\alpha(l,t)\{1-\alpha(l,t)\}^2]}{1-p^3(l,t)} + \frac{3p(l,t)[1-p(l,t)]^2[\alpha^2(l,t)+2\alpha(l,t)\{1-\alpha(l,t)\}]}{1-p^3(l,t)}+\frac{3\alpha(l,t)p^2(l,t)\{1-p(l,t)\}}{1-p^3(l,t)}, \tag{9}$$

$$\alpha_2(l+1,t)=\frac{[1-p(l,t)]^3[\alpha^3(l,t)+3\alpha^2(l,t)\{1-\alpha(l,t)\}]}{1-p^3(l,t)}+\frac{3p(l,t)[1-p(l,t)]^2[\alpha^2(l,t)+2\alpha(l,t)\{1-\alpha(l,t)\}]}{1-p^3(l,t)} + \frac{3\alpha(l,t)p^2(l,t)[1-p(l,t)]}{1-p^3(l,t)}, \tag{10}$$

$$\alpha_3(l+1,t)=\frac{[1-p(l,t)]^3\alpha^3(l,t)+3p(l,t)[1-p(l,t)]^2\alpha^2(l,t)}{1-p^3(l,t)}+\frac{3\alpha(l,t)p^2(l,t)[1-p(l,t)]}{1-p^3(l,t)}. \tag{11}$$

When concentrations of the mixture a_k, parameters of healing (β_0,c), and the appearance intensity (α_0) are fixed, then the system is governed by kinetic equations and the inverse cascade of appearance intensities. There is a stationary solution of this equations for $t\to\infty$. Densities of defects, densities of events, and appearance intensities of new defects tend to their limits when time grows: $p(l,t)\to p(l)$, $q(l,t)\to q(l)$, $\alpha(l,t)\to\alpha(l)$. Below we investigate scaling properties of the stationary solution $[p(l),q(l),\alpha(l)]$ for different parameters: concentrations of the mixture a_k and the appearance intensity of the bottom level α_0. Parameters of healing are also fixed, but their influence to the system behavior is estimated.

III. SYSTEM BEHAVIOR

Let us pass to limit for $t\to\infty$ in the kinetic equation (6). The limit density of defects $p(l)$ at level l satisfies the following equation:

$$p(l)\beta(l)=[1-p(l)]\alpha(l). \tag{12}$$

It means that the appearance of new defects and the healing of old defects compensate one another. Densities of defects and events may be expressed from Eqs. (8) and (12):

$$p(l)=\frac{\alpha(l)}{\alpha(l)+\beta(l)}, \tag{13}$$

$$q(l)=\frac{\alpha(l)\beta(l)}{\alpha(l)+\beta(l)}. \tag{14}$$

If $c<1$ then the healing intensity tends to zero for top levels of the system (2). In this case Eq. (14) means that the density of events $q(l)$ always tends to zero, when level l grows and events of high scale are less probable than small events. It is the most natural situation, so we mainly consider this parametric area $c<1$ with some remarks about system behavior under another conditions.

In contrast to the density of events, the density of defects $p(l)$ depends on the relation between appearance and healing intensities. Three kinds of the asymptotic behavior are possible when level l grows:

Stability. The appearance intensity $\alpha(l)$ tends to zero faster than the healing intensity $\beta(l)$. Then density of defects $p(l)$ also tends to zero, when level l grows $p(l)\to 0$. This kind of behavior is referred to as a *stability*, because top levels of the system remain unbroken.

Scale invariance. The appearance intensity $\alpha(l)$ tends to zero, similar to the healing intensity $\beta(l)$. In this case the density of defects $p(l)$ tends to a constant value different from zero and unity. This kind of behavior is referred to as a *scale invariance* because all scale levels of the system are similarly destroyed.

Catastrophe. The appearance intensity $\alpha(l)$ tends to a positive constant value or it tends to zero slower than the healing intensity $\beta(l)$. Then the density of defects $p(l)$ tends to unity. This kind of behavior is referred to as a *catastrophe*, because top levels of the system are completely destroyed. In this model catastrophic behavior is always realized for $\alpha \rightarrow$const, however, it is not obligatory in general case.

Let us consider Eq. (7) for $\alpha(l)$ close to unity. In first order of $[1-\alpha(l)]$ we obtain

$$\alpha_k(l+1)=\frac{[1-p(l)]^3\alpha(l)^3+3p(l)[1-p(l)]^2\alpha^2(l)+3p^2[1-p(l)]\alpha(l)}{1-p^3(l)}, \tag{15}$$

for all $k=1,2,3$. It follows from Eq. (7) that $1-p(l+1)$ is equal to zero, otherwise dividing by $1-p(l+1)$ we obtain that the appearance intensity $\alpha(l+1)$ also satisfies to Eq. (15). The former case means catastrophe; in the second case dividing by $\alpha(l)$ after simple transformation we obtain

$$\frac{\alpha(l+1)}{\alpha(l)}=\frac{[1+p(l)\{1-\alpha(l)\}]^2+[1+p(l)\{1-\alpha(l)\}]+p(l)}{1+p(l)+p^2(l)}. \tag{16}$$

It means that $\alpha(l+1)$ is greater than $\alpha(l)$, when $\alpha(l)$ is close to unity, therefore the catastrophic behavior always exists when the appearance intensity of the bottom level α_0 is big enough. It is not true for other kinds of behavior. Depending on concentrations of the mixture three kinds of behavior are realized when α_0 run from zero to unity.

Transition from stability to catastrophe. If $\alpha_0<\alpha_{cr}$ then densities of defects tend to zero, and if $\alpha_0>\alpha_{cr}$ then densities of defects tend to unity. In the critical point $\alpha_0=\alpha_{cr}$ densities of defects $p(l)$ tend to a constant value defined by Eq. (13) (Fig. 2). The scale invariance exists in a single point of phase transition.

Transition from scale invariance to catastrophe. If $\alpha_0 \leqslant \alpha_{cr}$ then densities of defects tend to a constant value different from zero and unity, and if $\alpha_0>\alpha_{cr}$ then densities of defects tend to unity (Fig. 3).

Total catastrophe. For all values of α_0 densities of defects $p(l)$ tend to unity (Fig. 4).

The critical point α_{cr} determines the boundary of catastrophe and depends on concentrations $a+k$ of the mixture and parameters of healing (Fig. 5).

IV. MAGNITUDE-FREQUENCY RELATION

The magnitude is a logarithmic measure of the energy of earthquakes. A linear relation between the earthquake magnitude of and the size of the earthquake source area is established in average [19],

$$\log_{10} S \approx M+\text{const.} \tag{17}$$

A linear relation between the logarithm of the number of earthquakes and their magnitudes, known as a Gutenberg-Richter law is established for the world seismicity [3] as well as for particular seismoactive regions [6],

$$\log_{10} N=a-bM. \tag{18}$$

In order to compare model result with seismicity we have to define the magnitude-frequency relation for the model. We assume that the size of elements in the system grows with level $S(l)=S_0 3^l$ and the number of elements similarly falls with level $N_e(l)=C3^{L-l}$ (C denotes number of elements at the highest level L of the system). Respecting the relation 17 between the magnitude of an earthquake and the linear size of its source area, we consider the magnitude as a characteristic of the defect's size at level l,

$$M(l)=l \log_{10} 3 \tag{19}$$

(we use decimal logarithm in respect to the geophysical tradition). Expressing the average number of events at the level l, as follows:

$$N(l)=C3^{L-l}q(l), \tag{20}$$

we obtain from Eqs. (20) and (19) the magnitude-frequency relation for events in our model, which is the analog of the Gutenberg-Richter law (18) for seismicity,

$$\log_{10} N(l)=-M(l)+\log_{10} q(l)+\text{const.} \tag{21}$$

It is obvious that the form of the magnitude-frequency relation is completely determined by the density of events $q(l)$, which depend on level l. A power form convergence of densities $q(l)$ to zero determines a linear form of the magnitude-frequency relation. Like the static mixture model [2], in general case, the dynamic system is characterized by linear form of the magnitude-frequency relation (Fig. 6). The slope of the magnitude-frequency relation is constant for areas of the scale invariance and the catastrophe areas; it depends on parameters of the mixture in the stability area

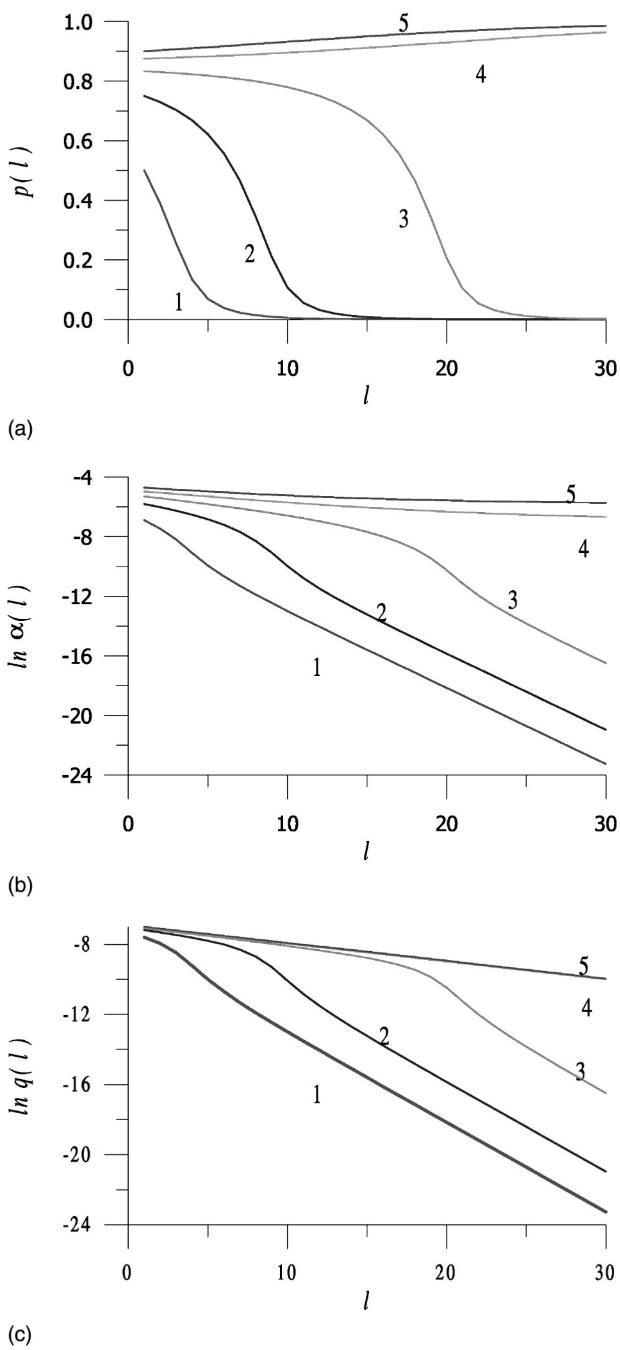

FIG. 2. Transition from stability to catastrophe: (a) density of defects; (b) logarithm of the appearance intensity; (c) logarithm of the density of events. Parameters of the mixture: $a_1=0.2$, $a_2=0$, $a_3=0.8$. Healing: $\beta_0=0.1$, $c=0.9$. Appearance intensity of the bottom level: $\alpha_0=0.1$ (curve 1); $\alpha_0=0.3$ (curve 2); $\alpha_0=0.5$ (curve 3); $\alpha_0=0.7$ (curve 4); $\alpha_0=0.9$ (curve 5).

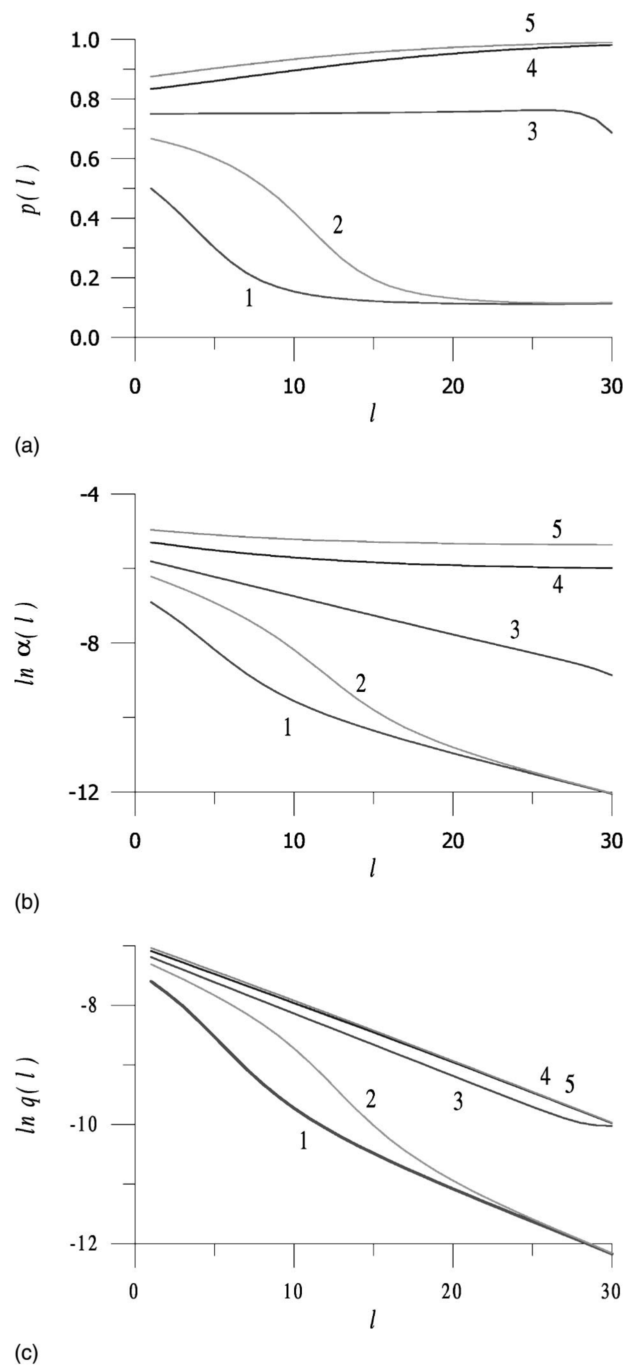

FIG. 3. Transition from scale invariance to catastrophe: (a) density of defects; (b) logarithm of the appearance intensity; (c) logarithm of the density of events. Parameters of the mixture: $a_1=0.4$, $a_2=0$, $a_3=0.6$. Healing: $\beta_0=0.1$, $c=0.9$. Appearance intensity of the bottom level: $\alpha_0=0.1$ (curve 1); $\alpha_0=0.2$ (curve 2); $\alpha_0=0.3$ (curve 3); $\alpha_0=0.5$ (curve 4); $\alpha_0=0.7$ (curve 5).

(Fig. 6). Below we express the slope of magnitude-frequency relation from equations determining the system behavior.

A. Area of scale invariance

In the area of the scale invariance the density of defects $p(l)$ tends to a constant value different from zero and unity and the appearance intensity $\alpha(l)$ tends to zero as c^l. Then Eq. (14) means, that the density of events also tends to zero as c^l. Substituting this approximation in the magnitude-frequency relation (21), we obtain the slope of the magnitude-frequency relation in the area of the scale invariance

$$b=1-\frac{\log_{10} c}{\log_{10} 3}. \tag{22}$$

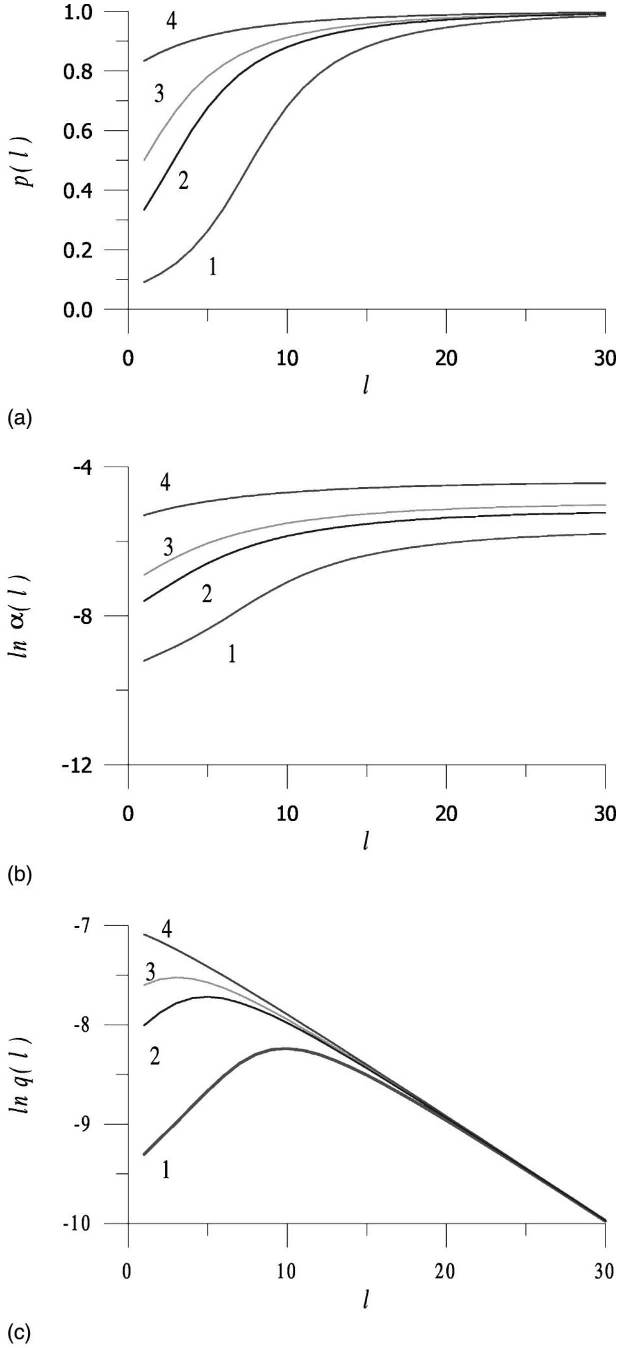

FIG. 4. Total catastrophe: (a) density of defects; (b) logarithm of the appearance intensity; (c) logarithm of the density of events. Parameters of the mixture: $a_1=0.4$, $a_2=0.5$, $a_3=0.1$. Healing: $\beta_0=0.1$, $c=0.9$. Appearance intensity of the bottom level: $\alpha_0=0.01$ (curve 1); $\alpha_0=0.05$ (curve 2); $\alpha_0=0.1$ (curve 3); $\alpha_0=0.5$ (curve 4).

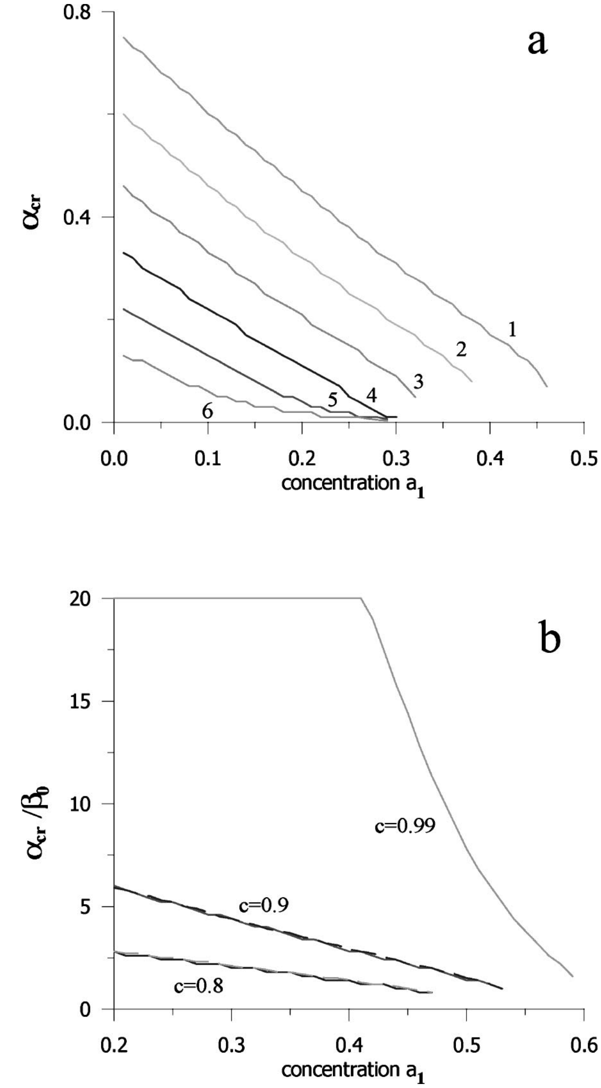

FIG. 5. Boundary of catastrophe α_{cr} vs concentration a_1. (a) Healing is fixed: $\beta_0=0.1$, $c=0.9$; concentrations a_2 are different: $a_2=0.1$ (curve 1), $a_2=0.2$ (curve 2), $a_2=0.3$ (curve 3), $a_2=0.4$ (curve 4), $a_2=0.5$ (curve 5), $a_2=0.6$ (curve 6). (b) $a_2=0$ is fixed, parameters of healing are $\beta_0=0.05$ (solid lines) and $\beta_0=0.1$ (dashed lines), c is indicated on the plot.

This slope is greater than unity when $c<1$; it is less than unity when $c>1$, and it is equal to unity only when $c=1$. Thus, the unity slope of the magnitude-frequency relation means that the scale invariance of the destruction $[p(l)=\text{const}]$ is complemented by the scale invariance of the healing $\beta(l)=\text{const}$.

B. Area of catastrophe

In the area of catastrophe the appearance intensity $\alpha(l)$ tends to a constant value, when level l grows. It follows from Eq. (13) that the density of unbroken elements may be expressed as follows:

$$1-p(l)=\frac{\beta(l)}{\alpha(l)+\beta(l)}. \tag{23}$$

Therefore the density of unbroken elements tends to zero as c^l, and Eq. (14) means that the density of events also tends to zero as c^l. This asymptotics is similar to the asymptotic in the area of the scale invariance, therefore, we obtain the same slope of the magnitude-frequency relation (22).

C. Area of stability

Let us consider Eqs. (9)–(11) for $t \to \infty$ under conditions $p \to 0$ and $\alpha \to 0$. Up to the first order of smallness, the appearance intensities $\alpha_k(l)$ are the following:

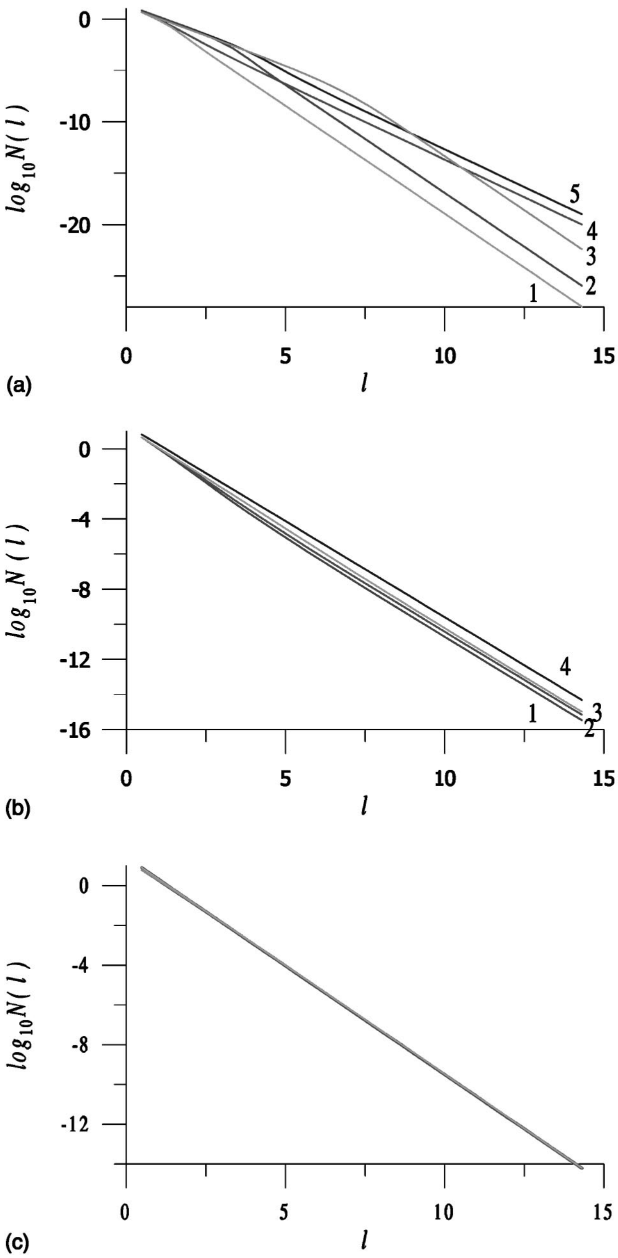

FIG. 6. Magnitude-frequency relation for different areas of system behavior: (a) stability; (b) scale invariance; (c) catastrophe. We consider levels $l=15,\ldots,30$; healing: $\beta_0=0.1$, $c=0.9$; other parameters: (a) $a_1=0.1$, $a_2=0.$, $a_3=0.9$, $\alpha_0=0.1$ (curve 1); $a_1=0.1$, $a_2=0.$, $a_3=0.9$, $\alpha_0=0.3$ (curve 2); $a_1=0.1$, $a_2=0.5$, $a_3=0.4$, $\alpha_0=0.1$ (curve 3); $a_1=0.2$, $a_2=0.$, $a_3=0.8$, $\alpha_0=0.1$ (curve 4); $a_1=0.2$, $a_2=0.$, $a_3=0.8$, $\alpha_0=0.3$ (curve 5). (b) $a_1=0.35$, $a_2=0$, $a_3=0.65$, $\alpha_0=0.1$ (curve 1); $a_1=0.4$, $a_2=0$, $a_3=0.6$, $\alpha_0=0.1$ (curve 2); $a_1=0.4$, $a_2=0.1$, $a_3=0.5$, $\alpha_0=0.1$ (curve 3); $a_1=0.4$, $a_2=0.$, $a_3=0.6$, $\alpha_0=0.3$ (curve 4); (c) Five curves coincided: $a_1=0.2$, $a_2=0$, $a_3=0.8$, $\alpha_0=0.7$; $a_1=0.4$, $a_2=0$, $a_3=0.6$, $\alpha_0=0.7$; $a_1=0.7$, $a_2=0$, $a_3=0.3$, $\alpha_0=0.7$; $a_1=0.7$, $a_2=0$, $a_3=0.3$, $\alpha_0=0.2$; $a_1=0.7$, $a_2=0.2$, $a_3=0.1$, $\alpha_0=0.2$.

$$\alpha_1(l)=3\alpha(l),$$
$$\alpha_2(l)=0, \tag{24}$$
$$\alpha_3(l)=0.$$

Substituting $\alpha_k(l)$ in Eq. (7) we obtain

$$\alpha(l+1)[1-p(l+1)]=3a_1\alpha(l)[1-p_1(l+1)]. \tag{25}$$

The density of defects $p_1(l)$ may be expressed through $\alpha_1(l)$ and $\beta(l)$, like Eq. (13), then

$$1-p_1(l+1)=\frac{\beta(l+1)}{\alpha_1(l+1)+\beta(l+1)}. \tag{26}$$

Substituting Eq. (26) in Eq. (25) and taking into account Eq. (24) we obtain

$$\alpha(l+1)[1-p(l+1)]=\frac{3a_1\alpha(l)(1-p(l))c(\alpha(l)+\beta(l))}{3\alpha(l)+\beta(l)c}. \tag{27}$$

The density of events is defined by Eq. (4), so we can change $\alpha(l)(1-p(l))$ to $q(l)$. Substituting $q(l)$ and dividing by $\beta(l)$ we obtain from Eq. (27) the following expression:

$$\frac{q(l+1)}{q(l)}=\frac{3a_1c(\varepsilon(l)+1)}{3\varepsilon(l)+c}, \tag{28}$$

where $\varepsilon=\alpha(l)/\beta(l)$ tends to zero in the area of the stability. Thus, we obtain the following approximation:

$$\frac{q(l+1)}{q(l)}=3a_1. \tag{29}$$

Consequently, the density of events $q(l)$ falls with level as $(3a_1)^l$. Substituting this approximation to the magnitude-frequency relation (21) we obtain the slope

$$b=-\frac{\log_{10}a_1}{\log_{10}3}. \tag{30}$$

In order to compare this slope with the slope of the magnitude-frequency relation in the area of the scale invariance it is necessary to estimate the concentration a_1 in the area of stability. Let us return to Eq. (25). Substituting the expression for $1-p_1(l+1)$ given by Eq. (26) and similar expression for $1-p(l+1)$, after some simple transformations we obtain

$$\frac{3a_1}{c}=\frac{\varepsilon(l+1)}{\varepsilon(l)}. \tag{31}$$

The right side is less than unity, thus we obtain the desired condition for concentrations of the mixture in the stability area

$$a_1<\frac{c}{3}. \tag{32}$$

Comparing slopes of the magnitude-frequency relation in the areas of stability, scale invariance and catastrophe, and respecting the condition (32), we obtain that the slope in the stability area is always greater than the slope in the area of the scale invariance or catastrophe. Similar result was previously obtained for the static mixed model [2].

V. CONDITIONS OF SCALE INVARIANCE

Let us determine what restriction have to be imposed to the mixture in order to obtain the area of the scale invariance. The scale invariance exists when the appearance intensity $\alpha(l)$ tends to zero like c^l, where c is the scaling coefficient of the healing intensity. The ratio $\alpha(l)/\beta(l)$ may be expressed from Eq. (13),

$$\frac{\alpha(l)}{\beta(l)}=\frac{p(l)}{1-p(l)}. \tag{33}$$

In first order of $\alpha(l)$ appearance intensities $\alpha_k(l+1)$ look as,

$$\alpha_1(l+1)=\frac{3\alpha(l)}{1+p(l)+p^2(l)}, \tag{34}$$

$$\alpha_2(l+1)=\frac{3\alpha(l)p(l)[2-p(l)]}{1+p(l)+p^2(l)},$$

$$\alpha_3(l+1)=\frac{3\alpha(l)p^2(l)}{1+p(l)+p^2(l)}.$$

Substituting Eq. (34) into Eq. (7) we obtain

$$\begin{aligned}\alpha(l+1)[1-p(l+1)]\\ &=\frac{3\alpha(l)}{1+p(l)+p^2(l)}[a_1\{1-p_1(l+1)\}+a_2p(l)[2-p(l)]\\ &\quad\times\{1-p_2(l+1)\}+a_3p^2(l)\{1-p_3(l+1)\}].\end{aligned} \tag{35}$$

Expressing $p_k(l+1)$ through $\alpha_k(l+1)$ and $\beta(l)$, dividing by $3\alpha(l)[1+p(l)+p^2(l)]$ and using Eq. (34) we can rewrite this equation as follows:

$$\begin{aligned}\frac{c(1+p(l)+p^2(l))[1-p(l+1)]}{3}&=[1-p(l+1)]-\frac{a_2c(1+p(l)+p^2(l))[1-p(l)]^2}{3\alpha(l)\beta(l)^{-1}p(l)[2-p(l)]+c(1+p(l)+p^2(l))}\\ &\quad-\frac{a_3c(1+p(l)+p^2(l))[1-p(l)^2]}{3\alpha(l)\beta(l)^{-1}p^2(l)+c(1+p(l)+p^2(l))}.\end{aligned} \tag{36}$$

Now we use the scale invariance $p(l+1)=p(l)=p$ and Eq. (33) for the ratio $\alpha(l)/\beta(l)$ and then obtain

$$\frac{c(1+p+p^2)}{3}=1-\frac{a_2c(1+p+p^2)(1-p)}{3p^2(2-p)(1-p)^{-1}+c(1+p+p^2)}-\frac{a_3c(1+p+p^2)(1+p)}{3p^3(1-p)^{-1}+c(1+p+p^2)}. \tag{37}$$

The value p is greater than zero. Substituting limits $p=0$ we obtain

$$a_1=\frac{c}{3}. \tag{38}$$

Comparing with the boundary of the stability area (32), we obtain the first condition of the scale invariance

$$a_1\geqslant c/3. \tag{39}$$

Now using this restriction we have

$$a_2\leqslant 1-\frac{c}{3}-a_3. \tag{40}$$

Substituting this boundary condition into Eq. (37) we obtain the restriction for a_3

$$a_3\geqslant D(p), \tag{41}$$

where

$$D(p)=\left[1-\frac{c(1+p+p^2)}{3}-\left(1-\frac{c}{3}\right)\frac{c(1-p^3)(1-p)}{3p^2(2-p)+c(1-p^3)}\right]\left(\frac{c(1-p^3)(1+p)}{3p^3+c(1-p^3)}-\frac{c(1-p^3)(1-p)}{3p^2(2-p)+c(1-p^3)}\right)^{-1}. \tag{42}$$

The function $D(p)$ is a monotone increasing function for p inside the interval $[0,1]$ (Fig. 7). Therefore, Eq. (41) means, that a_3 is greater than the minimum of $D(p)$ at $[0,1]$. The minimum is reached for $p=0$, so $a_3 \geqslant D(0)$. Considering $D(p)$ in the neighborhood of zero up to the first order of p we obtain the following restriction for a_3:

$$a_3 \geqslant \frac{1}{2} - \frac{c}{3}. \qquad (43)$$

In order to obtain the scale invariance, both conditions (39) and (43) have to be satisfied simultaneously, thus, like in the static model, the scale invariance exists only for highly heterogeneous conditions of the mixture.

VI. DISCUSSION AND CONCLUSIONS

We have investigated scaling properties of the stationary solution of the dynamical mixed model and we show that the critical behavior reflected in the linear form of the magnitude-frequency relation is a general case for DMM like it was for the static one [2]. Consequently this model may be considered as the self-organized critical system. The self-organized criticality in the dynamic model appears to be caused by the heterogeneity of the mixture — similar homogeneous model was not critical [18]. In order to obtain the scale invariance of defects the heterogeneity has to be strong enough. This result is important in order to understand the origin of the criticality of natural systems, in particular, of seismicity. The static model describe the destruction with infinite lifetime of defects and therefore it is more coincident to the description of laboratory experiments of sample destruction than to the modeling of the seismic process. The present dynamic model is not yet a perfect model of seismicity, however, it is more realistic than the static one and may be applied for modeling of average statistical properties of a stable seismic regime. Obtained results allow to suppose that the origin of criticality in seismicity could be related with heterogeneity of the lithosphere. Different parameters of the heterogeneity determine the difference of slopes of the Gutenberg-Richter law in different seismoactive regions.

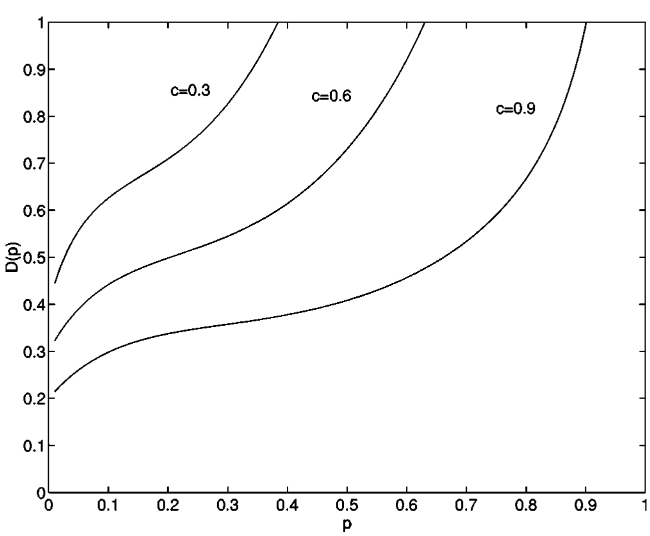

FIG. 7. Function $D(p)$ for different values of healing parameter c.

The static mixed model [1,2] may be interpreted as a particular special case of dynamic mixed model with zero or infinite lifetime of defects when $\beta(l)=1$ or $\beta(l)=0$ for all l. For both interpretations there is a scale invariance of healing, therefore, results relevant to the static model [2] are comparable with results obtained now for the dynamic one when $c=1$. It is possible to estimate the influence of the lifetime of defects (which value is inverse to the healing intensity) for the stationary system behavior. The introduction of a nontrivial healing brings to the appearance of the catastrophic behavior for any concentrations of the mixture when appearance intensity of the bottom level α_0 is big enough. Therefore pure cases of stability and scale invariance realized in the static model transform to the two-mode behavior: stability — catastrophe and scale invariance — catastrophe. Existence of the catastrophic area is quite natural and shows that the complete destruction of the system is always possible when the applied stress is big enough and that the system cannot be extremely resistant to the stress applied for a long time.

It is interesting that scaling parameter of healing c governs the slope of the magnitude-frequency relation in all areas of system behavior when parameters of the mixture satisfy to the stability area. The unity slope of the magnitude-frequency relation is really a very specific case of the critical behavior. It asks the scale invariance not only for the actual destruction [density of defects $p(l)$], but also for relaxing properties [healing $\beta(l)$] of the system. This fact attracts our attention to the problem of relaxation for the lithosphere of the Earth and its relation with critical properties of the seismic process.

Let us consider possible applications of obtained results to the problem of prediction of strong events. In the stability area the change of the appearance intensity α_0 is reflected in the change of the additive constant of the magnitude-frequency relation; the increasing instability to the applied stress is reflected in the slope of the magnitude-frequency relation (if a_1 grows) or also in the additive constant (if a_1 is constant and a_2 grows) [Fig. 6(a)]. Thus, in the area of stability any change of parameters that leads to the increasing of the probability of strong events is reflected in the change of the magnitude-frequency relation and a successful prediction is possible.

The slope of magnitude-frequency relation has the same constant value in both areas of the catastrophe and the scale invariance. The asymptotical behavior for the infinite size of the system $L \to \infty$ has a constant slope [Fig. 6(b) and (c)]. In the area of scale invariance the additive constant grows when the appearance intensity of the bottom level α_0 or the concentration of the mixture a_2 grow [Fig. 6(b)]; in the area of the catastrophe it does not change [Fig. 6(c)]. However there is a strong difference between the catastrophe and the scale invariance — the place of a future big event is well localized in the former case; the relative size of the risky area is close to zero [its probability is equal to $1-p(l)$]. This difference is important for a spatial prediction of strong events and allows to reduce the ratio of alarms in space.

We can conclude that the predictability may be different for different kinds of the system behavior although all these kinds demonstrate a linear magnitude-frequency relation. The slope of the magnitude-frequency relation reflects the change of system parameters only in the area of stability; the additive constant has some deviations in areas of stability and scale invariance. Therefore it is important to recognize what kind of system behavior is observed. Unlike static model [1,2], the slope of the magnitude-frequency relation cannot be considered as an indicator of the scale invariance, because it depends on the scaling parameter of healing c. However, if the healing parameter c could be estimated, the area of stability would be easy separated from that of scale invariance or catastrophe. It makes the study of relaxing properties of the system important for problems related with prediction of strong events, in particular, for the earthquake prediction.

ACKNOWLEDGMENTS

This publication was supported by a subcontract with Cornell University, Geological Sciences, under EAR-9804859 from the National Science Foundation and administrated by the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

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